A Berry Esseen result for the billiard transformation

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Abstract. *We consider billiard systems in the two dimensional torus with convex obstacles and finite horizon. In this paper, we prove a rate of convergence in $n^{-\frac{1}{2}}$ in the central limit theorem in the case of the billiard transformation. For one-dimensional functions, we control the maximal decay between the distribution functions. For multi-dimensional functions, we control the Prokhorov metric. This result gives some improvement to those of [13] and completes those of [14] in which the speed of convergence is envisaged in the sense of the Kantorovich metric. We use the construction of the Young towers [18] and Fourier calculations. A consequence of our result is a rate of convergence in $O(t^{-\frac{1}{4}+\alpha})$ in the central limit theorem for the billiard flow.*

1 Introduction

Let us fix some integer $\ell \geq 1$. We endow $\mathbb{R}^\ell$ with the supremum norm : $|(x_1, ..., x_\ell)|_\infty := \max_{i=1,...,\ell} |x_i|$. For any probability space $(\Omega, \mathcal{F}, \nu)$, any measurable space $(E, \mathcal{T})$ and any random variable $X : \Omega \to E$, we denote by $\nu_*(X)$ the image probability of ν by X, i.e. for all $A \in \mathcal{T}$, we have : $\nu_*(X)(A) = \nu(\{X \in A\}) = \nu(X^{-1}(A))$.

1.1 Central limit theorem, rate of convergence

Let a probability dynamical system $(\Omega, \mathcal{F}, \nu, T)$ be given, i.e. a probability space $(\Omega, \mathcal{F}, \nu)$ and a measurable transformation T of Ω preserving the probability measure ν. Let us fix some measurable function $f : \Omega \to \mathbb{R}^\ell$ with null expectation. We will use the following notations :

$$\forall \omega \in \Omega,\ S_0(f)(\omega) := 0_{\mathbb{R}^\ell} \ \text{ and } \ \forall n \geq 1,\ S_n(f,T)(\omega) := \sum_{k=0}^{n-1} f(T^k(\omega)).$$

We say that $((X_n := f \circ T^n)_{n\geq 0}, \nu)$ satisfies a central limit theorem if the sequence of random variables $\left(\frac{S_n(f,T)}{\sqrt{n}}\right)_{n\geq 1}$ converges in distribution to some (eventually generalised) gaussian variable Z. Once such a result established, it is natural to study the speed in this convergence. If $\ell = 1$ and if Z has nonnull variance, a classical way to do this is to control the following quantity :

$$DF_n := \sup_{x\in\mathbb{R}} \left| \nu\left(\frac{S_n(f,T)}{\sqrt{n}} \leq x\right) - \mathbb{P}(Z \leq x)\right|.$$

In higher dimension, we can estimate :

$$P_n := \Pi\left(\nu_*\left(\frac{S_n(f,T)}{\sqrt{n}}\right), \mathbb{P}_*(Z)\right),$$

where Π is the Prokhorov metric defined on the set of probability measures on $\mathbb{R}^\ell$.

1.2 Recalls about the Prokhorov metric

For a general reference on the Prokhorov metric, we refer to [6]. For any probability measures P and Q on $\mathbb{R}^\ell$, we define :

$$\Pi(P,Q) := \inf\left\{\varepsilon > 0 \ : \ \forall B \in \mathcal{B}(\mathbb{R}^\ell),\ (P(B) - Q(B^\varepsilon)) \le \varepsilon\right\}$$

where B^ε is the open ε-neighbourhood of B, i.e. $B^\varepsilon := \{x \in \mathbb{R}^\ell \ : \ \exists y \in \mathbb{R}^\ell \ \ |x-y|_\infty < \varepsilon\}$. With this definition, Π is a metric on the set of probability measures on $\mathbb{R}^\ell$. It is called the Prokhorov metric. Let us recall the link between the Prokhorov metric and the Ky-Fan metric $\mathcal{K}$. The Ky-Fan metric between two $\mathbb{R}^\ell$-valued random variables X and Y defined on a same probability space $(F, \mathcal{A}, p)$ is given by :

$$\mathcal{K}(X,Y) := \inf\{\varepsilon > 0 \ : \ p(|X-Y|_\infty > \varepsilon) < \varepsilon\}.$$

We recall that $\Pi(P,Q)$ corresponds to the infimum of the Ky-Fan metric $\mathcal{K}(X,Y)$ between two random variables X and Y defined on the same probability space such that X has the distribution P and such that Y has the distribution Q.

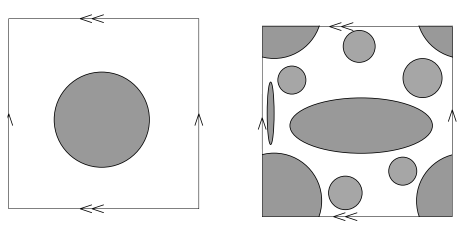
Let us notice that, if $\ell = 1$ and if Z has a nonnull variance, then a rate of convergence P_n in $O\left(n^{-\frac{1}{2}}\right)$ implies a rate of convergence DF_n in $O\left(n^{-\frac{1}{2}}\right)$. Let us recall that, if $(X_n)_{n\ge 0}$ is a sequence of independent identically distributed random variables admitting moments of the third order, the rate of convergence (in DF_n) is in $O(n^{-\frac{1}{2}})$ and it is optimal in the sense that there exists such a sequence $(X_n)_n$ for which DF_n is equivalent to $Cn^{-\frac{1}{2}}$ for some $C > 0$ (cf. [1, 7]).

1.3 Billiard transformation : definitions and results

We consider the billiard system in the two-dimensional torus with convex scatterers with finite horizon. Since the earliest article of Sinai [16] concerning ergodicity, a lot of articles have contributed to a better understanding of this system. The early proofs of central limit theorems in this context can be found in [4], [3]. A new approach is the method developped by Young in [18] extended by Chernov in [5] to the case of infinite horizon. We will use this aproach here.

Let us now introduce more precisely the system we are considering here. In the two-dimensional torus $\mathbb{T}^2$, we put a finite number of scatterers $O_1, ..., O_I$ (with $I \ge 1$) which are nonvoid, convex and open and the boundary of which is C^3. We also make the asumption that the closures of these sets are pairwise disjoint. The domain of our billiard is $Q := \mathbb{T}^2 \setminus \left(\bigcup_{i=1}^I O_i\right)$. Examples of such Q are given in figure I.

FIGURE I



We are interested in the behaviour of a point particle moving in Q with unit speed. We suppose that it respects the classical reflection law at each collision off a scatterer (reflected angle and incident angle are equal). We consider the billiard system (M, ν, T) corresponding to the times of collision :

- We consider the set $M := \bigcup_{i=1}^{I}\{i\} \times \frac{\mathbb{R}}{l_i\mathbb{Z}} \times \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$, where l_i is the length of ∂O_i. Let d_i be the metric induced on $\frac{\mathbb{R}}{l_i\mathbb{Z}}$ by the metric on $\mathbb{R}$ given by the absolute value. We endow M with a metric d such that, for all $i = \{1, ..., I\}$, for all $r, r' \in \frac{\mathbb{R}}{l_1\mathbb{Z}}$ and for all $\varphi, \varphi' \in \left]-\frac{\pi}{2};\frac{\pi}{2}\right[$, we have :

$$d((i,r,\varphi),(i,r',\varphi')) = \sqrt{d_i(r,r')^2 + |\varphi - \varphi'|^2}.$$

- The configuration $(i, r, \varphi) \in M$ corresponds to the collision of the scatterer O_i at the point q of ∂O_i of curvilinear absciss r and with an outgoing vector $\vec{v}$ making angle φ with the unit normal vector to ∂O_i at q oriented to the inside of Q (see figure II).

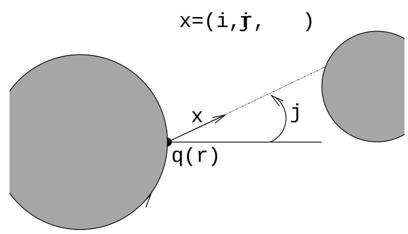


FIGURE II

- ν is the Borel probability measure on M proportional to $\sum_{i=1}^{I}(\cos(\varphi)\ dr\ d\varphi\delta_i)$.
- The billiard transformation T maps a configuration $x \in M$ of a particle at the time just after a collision off ∂Q) to the configuration $T(x) = x'$ of this particle at the time just after the next collision off ∂Q (see figure III).

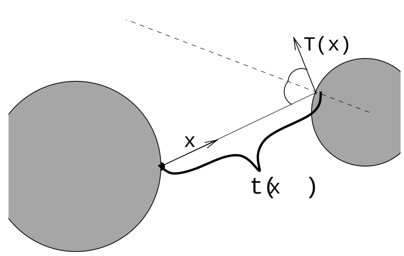


FIGURE III

We also define the function $\tau : M \to [0;+\infty[$ as follows : for all $x \in M$, $\tau(x)$ is the distance covered by a particle with configuration x until the next collision.

The billiard is said to have **finite horizon** if the function τ is uniformly bounded by some positive constant. It is said to have **infinite horizon** if τ is unbounded. In figure I, the first billiard domain has infinite horizon whereas the second one has finite horizon. **The results we state here hold for billiards with finite horizon.** The billiard transformation is discontinuous but it is regular on each one of its "continuous components". More precisely, let us define R_0 as the subset of M corresponding to vectors that are tangent to ∂Q :

$$R_0 := \left\{(i,r,\varphi) \in M \ : \ \varphi = \pm\frac{\pi}{2}\right\}.$$

For any $k \geq 1$, T^k defines a C^1-diffeomorphism from $M \setminus \bigcup_{j=0}^k (T^{-j}(R_0))$ onto $M \setminus \bigcup_{j=0}^k (T^j(R_0))$. In the finite horizon case, the sets $M \setminus \bigcup_{j=0}^k (T^{-j}(R_0))$ have a finite number of connected components.

Let us fix $\eta \in]0;1]$ and an integer $K \geq 0$. As in [13], we consider the set $\mathcal{H}_{(\eta,K)}$ of bounded functions $\phi : M \to \mathbb{R}$ such that the following quantity is finite :

$$C_\phi^{(\eta,K)} := \sup_{C \in \mathcal{C}_K} \sup_{x \in C, y \in C, x \neq y} \frac{|\phi(x)-\phi(y)|}{(\max(d(x,y),...,d(T^K(x),T^K(y))))^\eta},$$

where $\mathcal{C}_K$ is the set of connected component of $M \setminus \bigcup_{j=0}^K \left(T^{-j}(R_0)\right)$. Let $f \in \mathcal{H}_{(\eta,K)}$ be ν-centered. Because of exponential rate of decorrelation, we already know that the following limit exists :

$$\sigma^2(f) := \lim_{n \to +\infty} \mathbb{E}_\nu \left[\left(\frac{S_n(f,T)}{\sqrt{n}}\right)^2\right]$$

and is equal to : $\sigma^2(f) = \sum_{k \in \mathbb{Z}} \mathbb{E}_\nu \left[f.f \circ T^k\right]$. We also know that, if $\sigma^2(f) = 0$, then $(S_n(f,T))_n$ is bounded in $L^2(\nu)$.

In [13], we establish a rate of convergence $DF_n = O(n^{-\frac{1}{2}+\varepsilon})$ (for any $\varepsilon > 0$).

In [14], we establish a rate of convergence in $O(n^{-\frac{1}{2}})$ in the sense of Kantorovich (i.e. for the quantity $\sup_{\psi \in \mathcal{L}_1} \left|\mathbb{E}_\nu \left[\psi\left(\frac{S_n(f,T)}{\sqrt{n}}\right)\right] - \mathbb{E}[\psi(N)]\right|$, where $\mathcal{L}_1$ is the set of functions $\psi : \mathbb{R}^d \to \mathbb{R}$ such that, for all $x, y \in \mathbb{R}^d$, $|\psi(x)-\psi(y)| \leq |x-y|_\infty$). This rate is better than the previous one but we cannot deduce from it a result of convergence in $O(n^{-\frac{1}{2}})$ for DF_n or for P_n.

Theorem 1 (Onedimensional result) *We suppose that* $\sup \tau < +\infty$. *Let H be any function belonging to $\mathcal{H}_{(\eta,K)}$ with ν-expectation equal to 1 and such that* $\min_M H > 0$. *Let $f \in \mathcal{H}_{(\eta,K)}$ be ν-centered and such that $\sigma^2(f) > 0$. Then $\left(\frac{S_n(f,T)}{\sqrt{n}}\right)_n$ converges in distribution to a random variable Z with normal distribution $\mathcal{N}(0,\sigma^2(f))$ and there exists $A > 0$ such that we have*

$$\forall n \geq 1, \quad \sup_{x \in \mathbb{R}} \left|(H \cdot \nu)\left(\frac{S_n(f,T)}{\sqrt{n}} \leq x\right) - P(Z \leq x)\right| \leq \frac{A}{\sqrt{n}}.$$

A direct consequence of this is a central limit theorem for ℓ-dimensional f, the coordinates of which are all in $\mathcal{H}_{(\eta,K)}$ and ν-centered. The variance matrix of the limit random variable will be given by the following formula :

$$\Sigma^2(f) := \lim_{n \to +\infty} \mathbf{E}_\nu \left[\left(\frac{S_n(f,T)}{\sqrt{n}}\right)^{\otimes 2}\right]$$

with, for all $A, B \in \mathbb{R}^\ell$, $A \otimes B := (a_i b_j)_{i,j=1,\dots,\ell}$ and $A^{\otimes 2} := A \otimes A$.

Theorem 2 (Multidimensional result) *We suppose that* $\sup \tau < +\infty$. *Let H be any function belonging to $\mathcal{H}_{(\eta,K)}$ with ν-expectation equal to 1 and such that* $\min_M H > 0$. *Let $f : M \to \mathbb{R}^\ell$, the coordinates* *of which are in $\mathcal{H}_{(\eta,K)}$ and ν-centered. Then $\left(\frac{S_n(f,T)}{\sqrt{n}}\right)_n$ converges in distribution to a random variable Z with gaussian distribution $\mathcal{N}(0,\Sigma^2(f))$ and there exists $B>0$ such that we have*

$$\forall n\geq 1,\quad \Pi\left((H\cdot\nu)_*\left(\frac{S_n(f,T)}{\sqrt{n}}\right),\mathbb{P}_*(Z)\right)\leq \frac{B}{\sqrt{n}}.$$

Our proofs are based on the construction of the Young towers and on a perturbation method introduced by Nagaev in [11, 12] and adapted by many authors (cf. for example [9] and [10]). The results we used here come from [10]. The fact that our estimations hold for probability measures of the form $(H\cdot\nu)$ enables us to get an estimation in $O(t^{-\frac{1}{4}+\varepsilon})$ for any $\varepsilon>0$ in the case of the billiard flow (cf. the following section).

1.4 A consequence for the billiard flow

Here we suppose that billiard has finite horizon. Let us consider a particle moving with unit speed in the billiard domain Q. The configuration of such a particle at some time is given by a couple $(q,\theta)\in Q\times\mathbb{R}/\mathbb{Z}$ where q is the position of a particle and $2\pi\theta$ the angular measure between some fixed vector and the speed vector of the particle. To avoid confusion we eliminate configurations corresponding to incident vectors at the time of a collision off ∂Q. We denote by Q_1 the set obtained.

We define the billiard flow $(Z_t)_t$ as follows : $Z_t(q,\theta)=(q_t,\theta_t)$ is the configuration at time t of a particle with configuration (q,θ) at time 0. The flow $(Z_t)_t$ preserves the normalised Lebesgue measure μ_1 on Q_1.

The continuous system $(Q_1,(Z_t)_t,\mu_1)$ can be represented by the suspension flow $(\mathcal{M},(Y_t)_t,\mu)$ over (M,ν,T) with roof function τ. Indeed, Q_1 can be identified with $\mathcal{M}:=\{(x,s)\ :\ x\in M,\ s\in[0;\tau(x)[\}$ by $\psi:Q_1\to\mathcal{M}$ that maps $(q,\vec{v})\in Q_1$ to (x,s) where x is the configuration of the particle (presently at position q with speed $\vec{v}$) at the previous collision time and s is the distance covered since the previous collision. With this identification, Z_t corresponds to Y_t (i.e. $\psi\circ Z_t=Y_t\circ\psi$) where $Y_t(x,s)=(x,s+t)$ with the identifications $(y,\tau(x))=(T(y),0)$. Moreover the image probability measure of μ_1 by ψ is μ given by: $\mathbb{E}_\mu[g]=\int_M\left(\int_0^{\tau(x)}g(x,s)\,ds\right)d\nu(x)$.

Theorem 3 *We suppose that $\sup\tau<+\infty$. Let $F:Q_1\to\mathbf{R}^\ell$, the coordinates of which are η-Hölder continuous (for some $\eta\in]0;1]$) and μ_1 centered. Then the following limit exists : $\tilde{\Sigma}^2(F):=\lim_{t\to+\infty}\mathbf{E}_{\mu_1}\left[\left(\frac{1}{\sqrt{t}}\int_0^t F\circ Z_s\,ds\right)^{\otimes 2}\right]$ and $\left(\frac{1}{\sqrt{t}}\int_0^t F\circ Z_s\,ds\right)_t$ converges in distribution (when t goes to infinity) to a random variable W with gaussian distribution $\mathcal{N}(0,\tilde{\Sigma}^2(F))$ and for any $\varepsilon>0$, there exists $B_\varepsilon>0$ such that we have*

$$\forall t>1,\quad \Pi\left((\mu_1)_*\left(\frac{1}{\sqrt{t}}\int_0^t F\circ Z_s\,ds\right),\mathbb{P}_*(W)\right)\leq B_\varepsilon t^{-\frac{1}{4}+\varepsilon}.$$

In section 2, we explain how our results are linked with analogous results for the associated Young towers. Moreover we recall perturbation theorems used here. In section 3, we use the good property of the Young towers to prove theorem 1 and theorem 2. Let us mention that for the final step of the proof of theorem 1, another approach is given by the recent work of Gouëzel [8].

2 Preliminaries to the proofs

2.1 When $\Sigma^2(f)$ is non invertible

Let us consider that we are under the hypotheses of theorem 2. Let us suppose that $\Sigma^2(f)$ is non invertible. Then, up to a linear change of coordinates in $\mathbb{R}^\ell$, we suppose that the matrix $\Sigma^2(f)$ is a diagonal matrix with the first m diagonal terms equal to 1 and the others equal to zero. With this change, we have $Z = (Z_1, ..., Z_m, 0, ..., 0)$ and :

$$\Pi\left(\nu_*\left(\frac{S_n(f,T)}{\sqrt{n}}\right), \nu_*\left(\frac{S_n((f_1,...,f_m,0,...,0),T)}{\sqrt{n}}\right)\right) = O\left(\frac{1}{\sqrt{n}}\right) \qquad (1)$$

since we know that $\sigma^2(f_{m+j}) = 0$ implies that $(S_n(f_{m+j}, T))_n$ is bounded in $L^2(\hat{\nu})$.

If $m = 0$, the conclusion of theorem 2 is true according to formula (1).

If $m \geq 1$, then we are led to the study of the same problem with $(f_1, ..., f_m)$ instead of f.

Hence, in the proof of theorem 2, we suppose without any lost of generality that $\Sigma^2(f)$ is invertible.

2.2 Construction of the Young towers

Let us recall how Young constructs in [18] two dynamical systems $(\tilde{M}, \tilde{\nu}, \tilde{T})$ and $(\hat{M}, \hat{\nu}, \hat{T})$ such that :

- the system $(\tilde{M}, \tilde{\nu}, \tilde{T})$ is an extension of our billiard system (M, ν, T) and of the $(\hat{M}, \hat{\nu}, \hat{T})$, i.e. there exist two measurable functions $\pi : (\tilde{M}, \tilde{\nu}, \tilde{T}) \to (M, \nu, T)$ and $\hat{\pi} : (\tilde{M}, \tilde{\nu}, \tilde{T}) \to (\hat{M}, \hat{\nu}, \hat{T})$ such that : $\pi \circ \tilde{T} = T \circ \pi$, $\nu = (\pi)_* \tilde{\nu}$, $\hat{\pi} \circ \tilde{T} = \hat{T} \circ \hat{\pi}$ and $\hat{\nu} = (\hat{\pi})_* \tilde{\nu}$.
- the Perron-Frobenius operator P associated to $\hat{T}$ is quasi-compact on some good space of functions, and 1 is its only dominating eigenvalue (and it is simple).

Stable and unstable curves

We recall here some well known results about stable and unstable curves for (M, ν, T).

Definition *We call* **stable curve** *(resp.* **unstable curve***) a C^1-curve γ^s (resp. γ^u) of M contained in $M \setminus \bigcup_{k\geq 0} T^{-k} R_0$ (resp. in $M \setminus \bigcup_{k\geq 0} T^k R_0$) and satisfying $\lim_{n\to+\infty} l(T^n(\gamma^s)) = 0$ (resp. $\lim_{n\to+\infty} l(T^{-n}(\gamma^u)) = 0$), with $l(\gamma) := \int_\gamma \sqrt{dr^2 + d\varphi^2}$.*

Proposition 2.1 *There exists a set $\mathcal{M}$ of M, exactly T-invariant, such that $\nu(\mathcal{M}) = 1$ and such that any $x \in \mathcal{M}$ is contained in an unique maximal stable curve written $\gamma^s(x)$ and in an unique maximal unstable curve written $\gamma^u(x)$.*

Proposition 2.2 *There exist two real numbers $\alpha \in]0; 1[$ and $C > 0$ such that, for any stable curve γ^s, any unstable curve γ^u and any integer $n \geq 0$, we have $l(T^n(\gamma^s)) \leq C\alpha^n$ and $l(T^{-n}(\gamma^u)) \leq C\alpha^n$. Moreover, the intersection of a stable curve with an unstable curve contains at most one point.*

Construction of $(\tilde{M}, \tilde{\nu}, \tilde{T})$

Definition *We call* **rectangle** *of M a measurable subset A of M of the following form :*

$$A = \left(\bigcup_{\gamma^s \in \Gamma_A^s} \gamma^s\right) \cap \left(\bigcup_{\gamma^u \in \Gamma_A^u} \gamma^u\right),$$

where Γ_A^s is a family of stable curves and Γ_A^u a family of unstable curves and such that $\gamma^s \cap \gamma^u \neq \emptyset$, for any $(\gamma^s, \gamma^u) \in \Gamma_A^s \times \Gamma_A^u$.

*Let a rectangle A of M be given. We call s-***sub-rectangle** *of A a rectangle B of the following form :*

$$B = \left(\bigcup_{\gamma^s \in \Gamma_B^s} \gamma^s\right) \cap \left(\bigcup_{\gamma^u \in \Gamma_A^u} \gamma^u\right),$$

with Γ^s_B contained in Γ^s_A. We call u-**sub-rectangle** *of A a rectangle C of the following form :*

$$C = \left(\bigcup_{\gamma^s \in \Gamma^s_A} \gamma^s\right) \cap \left(\bigcup_{\gamma^u \in \Gamma^u_C} \gamma^u\right),$$

with Γ^u_C contained in Γ^u_A.

In [18], Young gives the construction of a rectangle $\Lambda = \left(\bigcup_{\gamma^s \in \Gamma^s} \gamma^s\right) \cap \left(\bigcup_{\gamma^u \in \Gamma^u} \gamma^u\right)$ contained in $\mathcal{M}$ (where Γ^s is a family of stable curves contained in $M \setminus T(R_0)$ and Γ^u a family of unstable curves contained in $M \setminus T^{-1}(R_0)$) endowed with a return time $R(\cdot)$ in Λ under the action of T and of a (countable) ν-essential partition $\{\Lambda_i\}_{i \geq 0}$ of Λ in s-sub-rectangles satisfying (in particular) the following :

- R is equal to a constant r_i on each Λ_i;
- For any $x \in \Lambda$, we have : $T^{R(x)}\left(\gamma^s(x)\right) \subseteq \gamma^s\left(T^{R(x)}(x)\right)$ and $T^{R(x)}\left(\gamma^u(x)\right) \supseteq \gamma^u\left(T^{R(x)}(x)\right)$.
- For any $i \geq 0$, $T^{r_i}(\Lambda_i)$ is a u-sub-rectangle of Λ;
- Λ_i is contained in a connected component of $M \setminus R_{-r_i,0}$.

Then, Young constructs a Borel probability measure $\tilde{\mu}$ on Λ, $T^{R(\cdot)}$-invariant, such that $\mathbf{E}_{\tilde{\mu}}[R] < +\infty$. The probability measure $\tilde{\mu}$ is a cluster value of $\left(\frac{1}{N}\sum_{k=0}^{N-1}\left(p_{\gamma^u_0}(\cdot|\gamma^u_0 \cap \Lambda)\right)_*\left((T^{R(\cdot)})^k\right)\right)_{N \geq 1}$ for the convergence in distribution (where γ^u_0 is some fixed unstable curve belonging to Γ^u and with $dp_\gamma = \mathbf{1}_\gamma \cos(\varphi)\, dr$). The dynamical system $\left(\tilde{M}, \tilde{\nu}, \tilde{T}\right)$ defined as follows is an extension of (M, ν, T) (by $\pi : \tilde{M}_1 \to M$ given by $\pi(x,l) = T^l(x)$) :

- $\tilde{M} := \{(x,l) : x \in \Lambda,\ 0 \leq l \leq R(x) - 1\}$;
- $\tilde{T}(x,l) = (x, l+1)$ if $l < R(x) - 1$ and $\tilde{T}(x,l) = \left(T^{R(x)}(x), 0\right)$ if $l = R(x) - 1$;
- $\tilde{\nu}\left(\bigcup_{l \geq 0} A_l \times \{l\}\right) = \frac{\sum_{l \geq 0} \tilde{\mu}(A_l)}{\mathbf{E}_{\tilde{\mu}}[R(\cdot)]}$, where, for each l, A_l is a measurable subset of $\{R > l\}$.

In the following, we suppose that $a := gcd(r_i)$ is equal to 1. We are led to this case :

- either by adapting Young's construction to make that true;
- or by noticing that a speed of convergence in the central limit theorem for $((f \circ T^n)_n, \nu)$ in $n^{-\frac{1}{2}}$ corresponds to establishing a speed of convergence in $n^{-\frac{1}{2}}$ for $\left((S_a(f,T) \circ T^{na})_n, \nu\right)$. If f is in $\mathcal{H}_{(\eta,K)}$, then $S_a(f,T)$ is in $\mathcal{H}_{(\eta,K+d-1)}$. Moreover, the system $(\tilde{M}_a, \tilde{\nu}_a, \tilde{T}_a)$ with $\tilde{M}_a := \bigcup_{l \geq 0} \Delta_{la}$, $\tilde{\nu}_a := a\left(\tilde{\nu}_{|\tilde{M}_a}\right)$ and $\tilde{T}_a := \left(\tilde{T}^a\right)_{|\tilde{M}_a}$ is an extension of (M, ν, T^a) that is analogous to extension $(\tilde{M}, \tilde{\nu}, \tilde{T})$ of (M, ν, T)

A partition

We define $i_l : \{x \in \Lambda : R(x) > l\} \to \Delta_l$ by $i_l(x) = (x,l)$. Young gives the construction of a partition $\mathcal{D} = \{\Delta_{l,j}; l \geq 0, j = 1, ..., j_l\}$ where $\{\Delta_{l,j}\}_j$ is a finite partition of $\Delta_l := \{(x,l') \in \tilde{M}; l' = l\}$ satisfying the following properties :

Properties 2.3 *1. $j_0 = 1$ and $\Delta_{0,1} = \Delta_0 = \Lambda \times \{0\}$;*

2. each ${i_l}^{-1}(\Delta_{l,j})$ is a s-sub-rectangle of Λ, union of Λ_i;

3. *For any $l \geq 0$, $\left\{ {i_{l+1}}^{-1}\left(\Delta_{l+1,j'}\right); j' = 1, ..., j_{l+1} \right\}$ is a partition of $\{R > l+1\}$ finer than the one induced by $\left\{ {i_l}^{-1}\left(\Delta_{l,j'}\right); j' = 1, ..., j_l \right\}$;*
4. *For any x, y in $i_l^{-1}(\Delta_{l,j})$ and in a same unstable curve, there exists an unstable curve containing x and y and contained in $M \setminus \bigcup_{k=0}^{l}(T^{-k}(R_0))$;*
5. *If $\tilde{T}^{-1}(\Delta_0) \cap \Delta_{l,j} \neq \emptyset$, then there exists an integer $i \geq 0$ such that $\tilde{T}^{-1}(\Delta_0) \cap \Delta_{l,j} = \Lambda_i \times \{r_i - 1\}$.*

For any $X, Y \in \tilde{M}_1$, we define the separation time $s(X,Y) := \max\left\{ n \geq 0 : \tilde{T}^n(Y) \in \mathcal{D}\left(\tilde{T}^n(X)\right) \right\}$.

Fact 2.4 *Let $n \geq 0$ be an integer. Let X and Y be two points in $\tilde{M}$ such that $s(X,Y) \geq n$. Then, the intersection point z of the curves $\gamma^s\left(\pi(X)\right)$ and $\gamma^u\left(\pi(Y)\right)$ exists. Moreover, $T^n(z)$ and $T^n\left(\pi(Y)\right)$ are both contained in a same unstable curve.*

A factor with a quasicompact transfer operator

We consider the factor $\left(\hat{M}, \hat{\nu}, \hat{T}\right)$ of $\left(\tilde{M}, \tilde{\nu}, \tilde{T}\right)$ given by the canonical projection $\hat{\pi} : \tilde{M} \to \hat{M}$, where $\hat{M}$ is the set of the $\mathcal{R}$-classes of $\tilde{M}$, for the binary relation $\mathcal{R}$ defined on $\tilde{M}$ by :

$$(x,l)\mathcal{R}(x',l') \quad \Leftrightarrow \quad l = l' \quad \text{and} \quad x, x' \text{ are in a same } \gamma^s \in \Gamma^s.$$

Young consider the measure $\hat{m}$ on $\hat{M}$ $\hat{m}(\hat{A}) = \sum_{l \geq 0} p_{\gamma_0^u}\left(\tilde{\pi}(\tilde{T}^{-l}(\hat{\pi}^{-1}(A) \cap \Delta_l))|\gamma_0^u \cap \Lambda\right)$ and prove that $\hat{\nu}$ is absolutely continuous relatively to $\hat{m}$ and such that the density $\hat{\rho} := \frac{d\hat{\nu}}{d\hat{m}}$ satisfies :

- ${c_0}^{-1} \leq \hat{\rho} \leq c_0$, for some real number $c_0 > 1$;
- $|\hat{\rho}(\hat{x}) - \hat{\rho}(\hat{y})| \leq c_1 \alpha_0^{\hat{s}(\hat{x},\hat{y})} \hat{\rho}(\hat{x})$, for some real numbers $c_1 > 0$ and $\alpha_0 \in]0;1[$;

with $\hat{s}(\hat{\pi}(x), \hat{\pi}(y)) := s(x,y)$. We shall write $\hat{\Delta}_l := \hat{\pi}\left(\Delta_l\right)$ and $\hat{\Delta}_{l,j} := \hat{\pi}\left(\Delta_{l,j}\right)$. Let us fix $\alpha_1 := \max(\alpha^{\frac{\eta}{2}}, \alpha_0)$. For any $\beta \in]0;1[$ and $\varepsilon > 0$, we define the functional space $\mathcal{V}_{(\beta,\varepsilon)}$ as follows :

$$\mathcal{V}_{(\beta,\varepsilon)} := \left\{ \hat{f} : \hat{M} \to \mathbb{C} \text{ measurable}, \ \|\hat{f}\|_{\mathcal{V}_{(\beta,\varepsilon)}} < +\infty \right\},$$

where $\left\|\hat{f}\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} := \left\|\hat{f}\right\|_{(\beta,\varepsilon,\infty)} + \left\|\hat{f}\right\|_{(\beta,\varepsilon,h)}$, with

$$\left\|\hat{f}\right\|_{(\beta,\varepsilon,\infty)} := \sup_{l \geq 0} \left\|\hat{f}_{|\hat{\Delta}_l}\right\|_\infty e^{-l\varepsilon},$$

$$\left\|\hat{f}\right\|_{(\beta,\varepsilon,h)} := \sup_{l \geq 0; j=1,\dots,j_l} \ \sup_{\hat{x},\hat{y} \in \hat{\Delta}_{l,j}} \frac{\left|\hat{f}(\hat{x}) - \hat{f}(\hat{y})\right|}{\beta^{\hat{s}(\hat{x},\hat{y})}} e^{-l\varepsilon}.$$

We define P as the adjoint operator of $g \mapsto g \circ \hat{T}$ on $L^2(\hat{m})$. We have $P\hat{\rho} = \hat{\rho}$. Young shows that we can find two real numbers $\beta \in]\alpha_1;1[$ and $\varepsilon_0 > 0$ such that, for any real number $\varepsilon \in]0;\varepsilon_0]$,

- There exists $C_0 > 0$ satisfying $\|\cdot\|_{L^2(\hat{\nu})} \leq C_0 \|\cdot\|_{\mathcal{V}_{(\beta,\varepsilon)}}$;
- There exist $\tau_1 \in]0;1[$ and $C_1 > 0$ such that, for any integer $n \geq 0$ and for any $\hat{f} \in \mathcal{V}_{(\beta,\varepsilon)}$ satisfying $\int_{\hat{M}} \hat{f}\, d\hat{m} = 0$, we have $\left\|P^n \hat{f}\right\|_{\mathcal{V}_{(\beta,\varepsilon)}} \leq C_1 {\tau_1}^n \left\|\hat{f}\right\|_{\mathcal{V}_{(\beta,\varepsilon)}}$;
- We have $P(\hat{f})(\hat{x}) = \sum_{\hat{z}:\hat{T}(\hat{z})=\hat{x}} \xi(\hat{z})\hat{f}(\hat{z})$, with $\left|\log \frac{\xi(\hat{x})}{\xi(\hat{y})}\right| \leq C_2 {\alpha_1}^{\hat{s}(\hat{x},\hat{y})-1}$, for any $\hat{x}$ and $\hat{y}$ in a same $\hat{\Delta}_{l,j}$.

In the following, we consider (β, ε) satisfying these properties. By a direct calculation, we get :

Lemma 2.5 *Let $h : \hat{M} \to \mathbb{C}$ be a bounded function such that there exists $c_h > 0$ such that :*

$$\forall \hat{x}, \hat{y} \in \hat{M}, \quad |h(\hat{x}) - h(\hat{y})| \leq c_h \beta^{\hat{s}(\hat{x},\hat{y})}.$$

Then, for any $g \in \mathcal{V}_{(\beta,\varepsilon)}$, function hg is in $\mathcal{V}_{(\beta,\varepsilon)}$ and we have :

$$||hg||_{(\beta,\varepsilon,\infty)} \leq ||h||_\infty ||g||_{(\beta,\varepsilon,\infty)} \text{ and } ||hg||_{(\beta,\varepsilon,h)} \leq ||h||_\infty ||g||_{(\beta,\varepsilon,h)} + c_h ||g||_{(\beta,\varepsilon,\infty)}.$$

2.3 From $f : M \to \mathbb{R}^\ell$ to $\hat{f} : \hat{M} \to \mathbb{R}^\ell$

Let us consider a function $f : M \to \mathbb{R}^\ell$ the coordinates of which are in $\mathcal{H}_{(\eta,K)}$ and are ν-centered.

First we define the function $\tilde{f} : \tilde{M} \to \mathbb{R}^\ell$ by $\tilde{f} := f \circ \pi$. Let us define $\tilde{H} := H \circ \tilde{\pi}$. The image measure $(\tilde{H} \cdot \tilde{\nu})_*(\tilde{\pi})$ coincide with $H \cdot \nu$. Hence, establishing a central limit theorem with a rate of convergence in $O(n^{-\frac{1}{2}})$ for $((f \circ T^n)_{n\geq 0}, H \cdot \nu)$ leads to establishing a central limit theorem with a rate of convergence in $O(n^{-\frac{1}{2}})$ for $((\tilde{f} \circ \tilde{T}^n)_{n\geq 0}, \tilde{H} \cdot \tilde{\nu})$.

Second we define $\hat{f} : \hat{M} \to \mathbb{R}^\ell$ such that we have : $\tilde{f} - \hat{f} \circ \hat{\pi} = h_f - h_f \circ \tilde{T}$ for some bounded function $h_f : \tilde{M} \to \mathbb{R}^\ell$. We follow the construction of [17] (this idea is already present in [2], lemma 1.6 pages 11-12). Let γ_0^u be some unstable curve of our rectangle Λ (such that $\gamma_0^u \cap \Lambda \neq \emptyset$). We define h_f as follows : $h_f := \sum_{n\geq 0} \left(\tilde{f} \circ \tilde{T}^n - \tilde{f} \circ \tilde{T}^n \circ \chi \right)$, where $\chi : \tilde{M} \to \{(x,l) \in \tilde{M} \ : \ x \in \gamma_0^u\}$ is the projection over the "images" of γ_0^u along the stable curves, that is : $\chi(x,l) = (x', l)$ where x' is the point of Λ belonging to γ_0^u and to the $\gamma^s \in \Gamma^s$ containing x. Because of exponential decay of the length of the stable curves, function h_f is well defined and bounded. We define $g_f : \tilde{M} \to \mathbb{R}^\ell$ as follows : $g_f := \tilde{f} - (h_f - h_f \circ \tilde{T})$. We have : $g_f := \tilde{f} \circ \chi + \sum_{n\geq 0} \left(\tilde{f} \circ \tilde{T}^{n+1} \circ \chi - \tilde{f} \circ \tilde{T}^n \circ \chi \circ \tilde{T} \right)$. This function g_f is constant along the stable curves. Therefore, we have : $g_f = \hat{f} \circ \hat{\pi}$ for some $\hat{f} : \hat{M} \to \mathbb{R}^\ell$ which is bounded (because $\tilde{f}$ and h are bounded). As in lemma 3.1 of [17], we prove the following :

Lemma 2.6 *There exists $c_{\hat{f}} > 0$ such that, for all $\hat{x}, \hat{y} \in \hat{M}$, we have $|\hat{f}(\hat{x}) - \hat{f}(\hat{y})|_\infty \leq c_{\hat{f}} \beta^{\hat{s}(\hat{x},\hat{y})}$.*

Proof. Let X and Y be two points of $\tilde{M}$.

(a) Let us control $\tilde{f}(\chi(X)) - \tilde{f}(\chi(Y))$. According to the fact 2.4, $T^{s(X,Y)}(\pi(\chi(X))$ and $T^{s(X,Y)}(\pi(\chi(Y))$ are in a same unstable curve. Therefore, for any $k = 0, ..., s(X,Y)$, we have :

$$d(T^k(\pi(\chi(X))), T^k(\pi(\chi(Y)))) \leq C\alpha^{s(X,Y)-k}.$$

If $s(X,Y) < K$, then we have : $\left|\tilde{f}(\chi(X)) - \tilde{f}(\chi(Y))\right|_\infty \leq \left(2||f||_\infty \alpha^{-\eta K}\right) \alpha^{\eta s(X,Y)}$.

If $s(X,Y) \geq K$, then we have : $\left|\tilde{f}(\chi(X)) - \tilde{f}(\chi(Y))\right| \leq C_f^{(\eta,K)} C^\eta \alpha^{-\eta K} \alpha^{\eta s(X,Y)}$.

(b) Let $n \geq 0$ be such that $2(n+K+1) \leq s(X,Y)$. Then $\pi\left(\tilde{T}^{s(X,Y)}(\chi(X))\right)$ and $\pi\left(\tilde{T}^{s(X,Y)}(\chi(Y))\right)$ belong to a same unstable curve. Hence we have :

$$\forall k = 0, ..., K, \quad d\left(T^k(\pi(\tilde{T}^{n+1}(\chi(X))), T^k(\pi(\tilde{T}^{n+1}(\chi(Y))))\right) \leq C\alpha^{s(X,Y)-(n+k+1)}$$

and so :

$$\left|\tilde{f}(\tilde{T}^{n+1}(\chi(X))) - \tilde{f}(\tilde{T}^{n+1}(\chi(Y)))\right|_\infty \leq C_f^{(\eta,K)} C^\eta \alpha^{\eta(s(X,Y)-(n+k+1))}.$$

Moreover, we have :

$$\forall k = 0, ..., K, \quad d\left(T^k(\pi(\tilde{T}^n(\chi(\tilde{T}(X)))), T^k(\pi(\tilde{T}^n(\chi(\tilde{T}(Y)))))\right) \leq C\alpha^{s(X,Y)-(n+k+1)}$$

and so :
$$\left|\tilde f(\tilde T^n(\chi(\tilde T(X)))) - \tilde f(\tilde T^n(\chi(\tilde T(Y))))\right|_\infty \le C_f^{(\eta,K)} C^\eta \alpha^{\eta(s(X,Y)-(n+k+1))}.$$

(c) Let $n \ge 0$ be such that $2(n+K+1) > s(X,Y)$. Let us notice that the points $\pi(\tilde T(\chi(X)))$ and $\pi(\chi(\tilde T(X)))$ are in a same stable curve. Hence we have :
$$\forall k=0,...,K, \quad d\left(T^k(\pi(\tilde T^n(\chi(\tilde T(X))))), T^k(\pi(\tilde T^{n+1}(\chi(X))))\right) \le C\alpha^n.$$
and therefore :
$$\left|\tilde f\left(\tilde T^n(\chi(\tilde T(X))\right) - \tilde f\left(\tilde T^{n+1}(\chi(X))\right)\right|_\infty \le C_f^{(\eta,K)} C^\eta \alpha^{\eta n}.$$
Analogously, we have :
$$\left|\tilde f\left(\tilde T^n(\chi(\tilde T(Y))\right) - \tilde f\left(\tilde T^{n+1}(\chi(Y))\right)\right|_\infty \le C_f^{(\eta,K)} C^\eta \alpha^{\eta n}.$$

(d) Conclusion :
$$\begin{aligned}|g_f(X) - g_f(Y)|_\infty \quad &\le \quad \left(2\|f\|_\infty + C_f^{(\eta,K)}C^\eta\right)\alpha^{-\eta K}\alpha^{\eta s(X,Y)} + \\ &\quad + 2C_f^{(\eta,K)}C^\eta \left(\sum_{n=0}^{\lfloor \frac{s(X,Y)}{2}\rfloor - K-1} \alpha^{\eta(s(X,Y)-(n+k+1))} + \sum_{n\ge \frac{s(X,Y)}{2}-K} \alpha^{\eta n}\right) \\ &\le \left(2\|f\|_\infty + C_f^{(\eta,K)}C^\eta\right)\alpha^{-\eta K}\alpha^{\eta s(X,Y)} + 2C_f^{(\eta,K)}C^\eta\left(\frac{\alpha^{\eta\left(\frac{s(X,Y)}{2}-1\right)}}{\alpha^{-\eta}-1} + \frac{\alpha^{\eta\left(\frac{s(X,Y)}{2}-K\right)}}{1-\alpha^\eta}\right),\end{aligned}$$

qed.

Lemma 2.7
$$\Pi\left((H\cdot\nu)_*\left(\frac{S_n(f,T)}{\sqrt n}\right), ((\tilde H\cdot\tilde\nu)_*(\hat\pi))_*\left(\frac{S_n(\hat f,\hat T)}{\sqrt n}\right)\right) \le \frac{2\|h_f\|_\infty}{\sqrt n}.$$

Therefore, to establish a rate of convergence in $n^{-\frac12}$ in the central limit theorem for $((f\circ T^n)_{n\ge0}, H\cdot\nu)$, it suffices to get a rate of convergence in $n^{-\frac12}$ in the central limit theorem for $\left((\hat f\circ\hat T^n)_{n\ge0}, (\tilde H\cdot\tilde\nu)_*(\hat\pi)\right)$. And to establish such a result, we can use the good properties of the operator $\hat P$ and apply the following results concerning techniques of perturbation of operators.

2.4 From $\tilde H\cdot\tilde\nu$ to $\hat H\cdot\hat\nu$

We will show that the image measure $(\tilde H\cdot\tilde\nu)_*(\hat\pi)$ is of the following form : $\hat H\cdot\hat\nu$ for some $\hat H$ belonging to $\mathcal V_{\beta,\varepsilon}$. Moreover we prove the existence of constants $a_H>0$, $b_H>0$ and $c_H>0$ such that :
$$\forall \hat x,\hat y\in\hat M, \quad a_H \le \hat H(\hat x)\le b_H, \quad |\hat H(\hat x) - \hat H(\hat y)| \le c_H\beta^{\hat s(\hat x,\hat y)}.$$
We follow the proof of the existence and of the properties of $\hat\rho$. We have :
$$\left((\tilde H\cdot\tilde\nu)_*(\hat\pi)\right)\left(\hat A\right) = (\tilde H\cdot\tilde\nu)\left(\bigcup_{l\ge0}\left(\left(\Lambda\cap\bigcup_{x\in\tilde A_l}\gamma^s(x)\right)\times\{l\}\right)\right),$$
with $\tilde A_l := \tilde\pi(\tilde T^{-l}(\hat\pi^{-1}(\hat A)\cap\Delta_l))$. According to the definitions of $\tilde\mu$ and of $\tilde\nu$, we have :
$$(\tilde H\cdot\tilde\nu)\left(\bigcup_{l\ge0}(B_l\times\{l\})\right) = \frac{1}{\mathbb E_{\tilde\mu}[R(\cdot)]}\sum_{l\ge0}(H\circ T^l\cdot\tilde\mu)(B_l).$$

Adapting the proof of lemma 2 of [18], we prove that, for all $l \geq 0$, there exists $g_l : \gamma_0^u \cap \Lambda \to \mathbb{R}$ such that :

$$(H \circ T^l \cdot \tilde{\mu}) \left(\bigcup_{x \in \tilde{A}_l} \gamma^s(x) \right) = (g_l \cdot p_{\gamma_0^u}(\cdot | \gamma_0^u \cap \Lambda))(\tilde{A}_l),$$

with $\tilde{A}_l$ any measurable subset of $\gamma_0^u \cap \Lambda$. Moreover there exist $a_h > 0$, $b_h > 0$ and $c_h > 0$ such that

$$\forall l \geq 0,\ \forall x \in \gamma_0^u \cap \Lambda, \quad a_h \leq g_l(x) \leq b_h$$

and :

$$\forall l \geq 0,\ \forall x, y \in \gamma_0^u \cap \Lambda, \quad |g_l(x) - g_l(y)| \leq c_h g_l(x) \beta^{\frac{s((x,0),(y,0))-l}{2}}.$$

From this and from the definition of $\hat{m}$ we conclude that $(\tilde{H} \cdot \tilde{\nu})_*(\hat{\pi})$ is absolutely continuous with respect to $\hat{m}$ and we have :

$$\frac{d((\tilde{H} \cdot \tilde{\nu})_*(\hat{\pi}))}{d\hat{m}}(\hat{\pi}(x,l)) = \frac{g_l(x)}{\mathbb{E}_{\tilde{\mu}}[R(\cdot)]}.$$

Now, let us indicate how we prove the existence of the functions g_l. Let $\tilde{A}_l$ be any measurable subset of $\gamma_0^u \cap \Lambda$. Let $(N_k)_{k \geq 1}$ be a sequence of integers such that $\left(\frac{1}{N_k} \sum_{j=0}^{N_k - 1} \left(p_{\gamma_0^u}(\cdot | \gamma_0^u \cap \Lambda) \right)_* ((T^{R(\cdot)})^j) \right)_{k \geq 1}$ converges in distribution to $\tilde{\mu}$. We have :

$$\begin{aligned}
(H \circ T^l \cdot \tilde{\mu}) \left(\bigcup_{x \in \tilde{A}_l} \gamma^s(x) \right) &= \lim_{k \to +\infty} \frac{1}{N_k} \sum_{j=0}^{N_k - 1} \left(H \circ T^l \cdot p_{\gamma_0^u}(\cdot | \gamma_0^u \cap \Lambda) \right)_* ((T^{R(\cdot)})^j) \left(\bigcup_{x \in \tilde{A}_l} \gamma^s(x) \right) \\
&= \lim_{k \to +\infty} \int_{\tilde{A}_l} H_{N_k, l} \, dp_{\gamma_0^u \cap \Lambda}(\cdot | \gamma_0^u \cap \Lambda),
\end{aligned}$$

with $H_{N,l}(y) := \frac{1}{N} \sum_{j=0}^{N-1} \sum_{\mathbf{i} = (i_0, \ldots, i_{j-1}) \in \mathbb{N}^j} \rho_{j,\mathbf{i}}(y) H(T^l(y_{j,\mathbf{i}}))$, where $y_{j,\mathbf{i}}$ is the intersection point belonging to $\gamma^s(y)$ and to the unstable curve $\gamma_{(\mathbf{i})}^u \in \Gamma^u$ containing the following set :

$$(T^{R(\cdot)})^j \left(\gamma_u^0 \cap \bigcap_{m=0}^{j-1} \left(T^{R(\cdot)} \right)^{-m} (\Lambda_{i_m}) \right)$$

and with :

$$\rho_{j,\mathbf{i}}(y) := \left(\prod_{m \geq 0} \frac{J^u T(T^m(y))}{J^u T(T^m(y_{j,\mathbf{i}}))} \right) \frac{1}{\prod_{k=0}^{r_{i_0} + \ldots + r_{i_{j-1}} - 1} J^u T(T^k(y_{j,\mathbf{i}}))},$$

with $J^u T(x) = \frac{d((p_{\gamma^u(x)})_*(T))}{dp_{\gamma^u_{T(x)}}}(x)$. According to the properties satisfied by J^u (cf. [18]), there exists $K > 0$ such that :

$$\forall j,\ \forall \mathbf{i},\ \forall x, y \in \gamma_0^u \cap \Lambda, \quad \left| \log \frac{\rho_{j,\mathbf{i}}(x)}{\rho_{j,\mathbf{i}}(y)} \right| \leq K \beta^{s((x,0),(y,0))}.$$

Moreover, we have :

$$\forall x, y \in \gamma_0^u \cap \Lambda, \quad \left| \log \frac{H(T^l(x_{j,\mathbf{i}}))}{H(T^l(y_{j,\mathbf{i}}))} \right| \leq \frac{1}{\min_M H} \left(C_H^{(\eta,K)} + 2\|H\|_\infty \right) C^\eta \alpha^{\eta(s((x,0),(y,0)) - l - K)}.$$

Let us write $\mathcal{L}_H := K + \frac{1}{\min_M H} \left(C_H^{(\eta,K)} + 2\|H\|_\infty \right) C^\eta \alpha^{-\eta K}$. We have :

$$\forall x, y \in \gamma_0^u \cap \Lambda, \quad \left| \log \frac{\rho_{j,\mathbf{i}}(x) H(T^l(x_{j,\mathbf{i}}))}{\rho_{j,\mathbf{i}}(y) H(T^l(y_{j,\mathbf{i}}))} \right| \leq \mathcal{L}_H \beta^{s((x,0),(y,0)) - l}.$$

From which we get :

$$\forall x, y \in \gamma_0^u \cap \Lambda, \quad |H_{N,l}(x) - H_{N,l}(y)| \leq \mathcal{L}_H e^{\mathcal{L}_H} H_{N,l}(x) \beta^{s((x,0),(y,0)) - l}.$$

From the Ascoli theorem, we get the uniform convergence of some subsquence $(H_{N_{k_m}, l})_m$ of $(H_{N_k, l})_k$ to some function g_l satisfying the properties said before.

2.5 Speed of convergence and characteristic functions

Let us recall the two following results linking speed of convergence in the sense of DF_n and of P_n to estimates on the characteristic functions.

For any real random variable X, we denote by $\varphi_X : \mathbb{R} \to \mathbb{C}$ the characteristic function of X, i.e. $\varphi_X : t \mapsto \mathbb{E}[e^{itX}]$.

Theorem 2.8 (Berry-Esseen lemma, $\ell = 1$) *Let Y be a random variable with gaussian law (with non-null variance). For any real random variable X and for any real number $U > 0$, we have :*

$$\sup_{x \in \mathbb{R}} |\mathbb{P}(X \le x) - \mathbb{P}(Y \le x)| \le \frac{1}{\pi} \int_{-U}^{U} \frac{|\varphi_X(t) - \varphi_Y(t)|}{|t|}\, dt + \frac{24}{\pi\sqrt{2\pi}U}.$$

We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product on $\mathbb{R}^\ell$. For any $\mathbb{R}^\ell$-random variable X, we denote by $\varphi_X : \mathbb{R}^\ell \to \mathbb{C}$ the characteristic function of X, i.e. $\varphi_X : t \mapsto \mathbb{E}[e^{i\langle t, X\rangle}]$.

Theorem 2.9 (Yurinskii [19], $\ell \ge 1$) *Let Y be some ℓ-dimensional random variable with gaussian distribution (with invertible covariance matrix). There exist two real numbers $c_0 > 0$ and $\Gamma > 0$ such that, for any real number $T > 0$ and for any $\mathbb{R}^\ell$-random variable X (defined on some $(E, \mathcal{T}, \mathbb{P})$), we have :*

$$\Pi(\mathbb{P}_*(X), \mathbb{P}_*(Y)) \le c_0 \left(\frac{1+\Gamma}{T} + \left(\int_{|t|_\infty < T} \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor + 1} \sum_{\{i_1,\dots,i_k\} \in \{1,\dots,\ell\}^k} \left| \frac{\partial^k}{\partial t_{i_1} \cdots \partial t_{i_k}} (\varphi_X - \varphi_Y)(t) \right|^2 \right)^{\frac{1}{2}} \right).$$

Hence, to establish theorems 1 and 2, we can use estimates on characteristic functions.

2.6 Characteristic functions and operators

The adjoint operator $\hat{P}$ of $g \mapsto g \circ \hat{T}$ on $L^2(\hat{\nu})$ is given by : $\hat{P}(g) := \frac{P(\hat{\rho} g)}{\hat{\rho}}$, where P is the adjoint operator of $g \mapsto g \circ \hat{T}$ on $L^2(\hat{m})$. Because of the properties of P and of $\hat{\rho}$ (and according to lemma 2.5), the operator $\hat{P}$ satisfies the following property :

- $\hat{P}$ is a continuous linear operator on $\mathcal{V}_{(\beta,\varepsilon)}$;
- we have $\hat{P}\mathbf{1} = \mathbf{1}$;
- there exist two real numbers $C_2 > 0$ and $\tau_2 \in]0;1[$ such that, for all integer $n \ge 0$ and for all $\hat{f} \in \mathcal{V}_{(\beta,\varepsilon)}$ such that $\int_{\hat{M}} \hat{f}\, d\hat{\nu} = 0$, we have : $\left\| \hat{P}^n \hat{f} \right\|_{\mathcal{V}_{(\beta,\varepsilon)}} \le C_2 {\tau_2}^n \left\| \hat{f} \right\|_{\mathcal{V}_{(\beta,\varepsilon)}}$.

We will use the fact that we have : $\mathbb{E}_{\hat{H}\cdot\hat{\nu}}\left[e^{\frac{i\langle t, S_n(\hat{f},\hat{T})\rangle}{\sqrt{n}}} \right] = \mathbb{E}_{\hat{\nu}}\left[\left(\hat{P}_{\frac{t}{\sqrt{n}}} \right)^n (\hat{H}) \right]$ with :

$$\hat{P}_t(g) = \hat{P}(e^{i\langle t, \hat{f}\rangle} g). \tag{2}$$

Indeed, it is easy to see that, for all $t \in \mathbb{R}^\ell$ and $n \ge 1$, we have : $(\hat{P}_t)^n(g) = \hat{P}^n \left(e^{i\langle t, S_n(\hat{f},\hat{T})\rangle} g \right)$. We will apply the theorems of perturbation recalled in the following section to our $(\hat{P}_t)_t$. Let us introduce a few notations : $\mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}}$ will be the set of continuous $\mathbb{C}$-linear operator of $\mathcal{V}_{(\beta,\varepsilon)}$. We endow $\mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}}$ with the norm $\|\cdot\|_{\mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}}}$ given by : $\|\Psi\|_{\mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}}} := \sup_{\|f\|_{\mathcal{V}_{(\beta,\varepsilon)}}=1} \|\Psi(f)\|_{\mathcal{V}_{(\beta,\varepsilon)}}$.

Lemma 2.10 *The map $t \mapsto \hat{P}_t$ is in $C^\infty(\mathbb{R}^\ell, \mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}})$. Moreover, for all $t \in \mathbb{R}^\ell$, for all $m \geq 1$ and all $i_1, ..., i_m \in \{1, ..., \ell\}$, we have :*

$$\frac{\partial^m}{\partial t_{i_1} \cdots \partial t_{i_m}} \hat{P}_t(g) = \hat{P}_t(\hat{f}_{i_1} \cdots \hat{f}_{i_m} g) = \hat{P}\left(i^m e^{i\langle t, \hat{f}\rangle} \hat{f}_{i_1} \cdots \hat{f}_{i_m} g\right).$$

Proof.

- First of all, we notice that, for any $t \in \mathbb{R}^\ell$, $e^{i\langle t, \hat{f}\rangle}$ satisfies the hypothesis of lemma 2.5. Therefore, for any $t \in \mathbb{R}^\ell$, $\hat{P}_t$ belongs to $\mathcal{L}_{\mathcal{V}_{\beta,\varepsilon}}$.

- Let $t \in \mathbb{R}^\ell$, $v \in \mathbb{R}^*$, $j \in \{1, ..., \ell\}$ and $g \in \mathcal{V}_{\beta,\varepsilon}$. Let us prove that we have :

$$\left\| \frac{\hat{P}_{t+ve_j}(\cdot) - \hat{P}_t(\cdot) - v\hat{P}_t(i\hat{f}_j \times \cdot)}{v} \right\|_{\mathcal{L}_{\mathcal{V}_{\beta,\varepsilon}}} \leq \|\hat{P}_t\|_{\mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}}} \left(\frac{|v|}{2} \|\hat{f}\|_\infty^2 + 2|v| c_{\hat{f}} \|\hat{f}\|_\infty \right),$$

  where e_j is the j^{th} vector of the canonical basis of $\mathbb{R}^\ell$. We have :

$$\frac{\hat{P}_{t+ve_j}(g) - \hat{P}_t(g) - v\hat{P}_t(i\hat{f}_j g)}{v} = \hat{P}_t\left(\frac{e^{iv\hat{f}_j} - 1 - i\hat{f}_j v}{v} g \right).$$

  Hence, we have :

$$\left\| \frac{\hat{P}_{t+ve_j}(g) - \hat{P}_t(g) - v\hat{P}_t(i\hat{f}_j g)}{v} \right\|_{\mathcal{V}_{(\beta,\varepsilon)}} \leq \left\| \hat{P}_t \right\|_{\mathcal{L}_{\mathcal{V}_{(\beta,\varepsilon)}}} \left\| \frac{e^{iv\hat{f}_j} - 1 - i\hat{f}_j v}{v} g \right\|_{\mathcal{V}_{(\beta,\varepsilon)}}.$$

  We will apply lemma 2.5 to the function $h := \frac{e^{iv\hat{f}_j} - 1 - i\hat{f}_j v}{v}$. Indeed, we have :

$$\|h\|_\infty = \left\| \frac{e^{iv\hat{f}_j} - 1 - i\hat{f}_j v}{v} \right\|_\infty \leq \frac{|v|}{2} \|\hat{f}\|_\infty^2$$

  and, for any $\hat{x}$ and $\hat{y}$ in $\hat{M}$, we have :

$$\begin{aligned} |h(\hat{x}) - h(\hat{y})| &= \left| \frac{e^{iv\hat{f}_j(\hat{x})} - e^{iv\hat{f}_j(\hat{x}) + iv(\hat{f}_j(\hat{y}) - \hat{f}_j(\hat{x}))} - i(\hat{f}_j(\hat{x}) - \hat{f}_j(\hat{y}))v}{v} \right| \\ &\leq \left| \frac{(e^{iv\hat{f}_j(\hat{x})} - 1) iv(\hat{f}_j(\hat{x}) - \hat{f}_j(\hat{y}))}{v} \right| + \frac{v^2 |\hat{f}_j(\hat{y}) - \hat{f}_j(\hat{x})|^2}{2|v|} \\ &\leq 2|v|.\|\hat{f}\|_\infty \left| \hat{f}_j(\hat{x}) - \hat{f}_j(\hat{y}) \right|. \end{aligned}$$

- Conclusion. We have :

$$\lim_{v \to 0} \left\| \frac{\hat{P}_{t+ve_j}(\cdot) - \hat{P}_t(\cdot) - v\hat{P}_t(\hat{f}_j \times \cdot)}{v} \right\|_{\mathcal{L}_{\mathcal{V}_{\beta,\varepsilon}}} = 0$$

  and, for all $m \geq 1$ and all $i_1, ..., i_m, j \in \{1, ..., \ell\}$, we have :

$$\lim_{v \to 0} \left\| \frac{\hat{P}_{t+ve_j}(i^m \hat{f}_{i_1} \cdots \hat{f}_{i_m} \times \cdot) - \hat{P}_t(i^m \hat{f}_{i_1} \cdots \hat{f}_{i_m} \times \cdot) - v\hat{P}_t(i^{m+1} \hat{f}_j \hat{f}_{i_1} \cdots \hat{f}_{i_m} \times \cdot)}{v} \right\|_{\mathcal{L}_{\mathcal{V}_{\beta,\varepsilon}}} = 0,$$

qed

2.7 Perturbation operator method

First let us introduce some notations. For any complex Banach space $\mathcal{B}$, we use the following notations :

1. We denote by $\mathcal{B}'$ the set of continuous $\mathbb{C}$-linear maps from $\mathcal{B}$ in $\mathbb{C}$. We endow this set of the norm $\|\cdot\|_{\mathcal{B}'}$ given by : $\|A\|_{\mathcal{B}'} := \sup_{\|f\|_{\mathcal{B}}=1} |A(f)|$.
2. For any $A \in \mathcal{B}'$ and any f in $\mathcal{B}$, we will use the notation : $\langle A, f\rangle_* := A(f)$.
3. For any $A \in \mathcal{B}'$, any $g \in \mathcal{B}$, we denote by $g \otimes_* A$ the continuous $\mathbb{C}$-linear endomorphism of $\mathcal{B}$ defined by : $(g \otimes_* A)(f) := \langle A, f\rangle_* g$.
4. We denote by $\mathcal{L}_{\mathcal{B}}$ the set of continuous $\mathbb{C}$-linear endomorphisms of $\mathcal{B}$. We endow this set with the norm $\|\cdot\|_{\mathcal{L}_{\mathcal{B}}}$ given by : $\|P\|_{\mathcal{L}_{\mathcal{B}}} := \sup_{\|f\|_{\mathcal{B}}=1} \|P(f)\|_{\mathcal{B}}$.

Theorem 2.11 (theorem III.8 of [10], Onedimensional version) *Let $\mathcal{B}$ be a complex Banach space. Let I_0 be an open interval containing 0. Let $m \geq 1$ be some integer. Let $(Q(t))_{t\in I_0}$ be a family of continuous linear operators on $\mathcal{B}$ such that :*

(i) The application $t \mapsto Q(t)$ is in $C^m(I_0, \mathcal{L}_{\mathcal{B}})$;

(ii) there exist two subspaces $\mathcal{F}$ and $\mathcal{H}$ of $\mathcal{B}$ such that :

(a) $\mathcal{B} = \mathcal{F} \oplus \mathcal{H}$, $Q(0)(\mathcal{F}) \subseteq \mathcal{F}$ and $Q(0)(\mathcal{H}) \subseteq \mathcal{H}$.

(b) $\dim(\mathcal{F}) = 1$ and $Q(0)_{|\mathcal{F}} \equiv id_{|\mathcal{F}}$,

(c) the spectral radius of $Q(0)_{|\mathcal{H}}$ is strictly less than 1.

Then there exists an open interval I_1 containing 0 and contained in I_0, there exist real numbers $\eta_1 > 0, \eta_2 > 0$, $c_1 \geq 0$ and four functions $\lambda \in C^m(I_1, \mathbb{C})$, $v \in C^m(I_1, \mathcal{B})$, $\varphi \in C^m(I_1, \mathcal{B}')$ and $N \in C^m(I_1, \mathcal{L}_{\mathcal{B}})$ such that, for all $t \in I_1$, we have :

(1) for all $n \geq 1$, $Q(t)^n = \lambda(t)^n v(t) \otimes_* \varphi(t) + N(t)^n$,

(2) $Q(t)v(t) = \lambda(t)v(t)$, $Q(t)^*\varphi(t) = \lambda(t)\varphi(t)$ and $\langle\varphi(t), v(t)\rangle_* = 1$,

(3) $|\lambda(t)| \geq 1 - \eta_1$,

(4) for all $\ell = 0, ..., m$ and all $n \geq 1$, $\|\frac{d^\ell}{dt^\ell} N(t)^n\|_{\mathcal{L}_{\mathcal{B}}} \leq c_1(1 - \eta_1 - \eta_2)^n$.

(5) [Corollaries III-11 and III-12 of [10]] Moreover, if $m \geq 2$, we have :

(a) $\lambda'(0) = \langle\varphi(0), Q'(0) \cdot v(0)\rangle_*$

(b) and, if $\lambda'(0) = 0$, $\sup_{n\geq 0} |n\lambda''(0) - \langle\varphi(0), (Q^n)''(0) \cdot v(0)\rangle_*| < +\infty$.

We will use this theorem to prove our theorem 1 (when $\ell = 1$)with $Q(t) = \hat{P}_t$ and $\mathcal{B} = \mathcal{V}_{(\beta,\varepsilon)}$ and $Q^{(m)}(t) \cdot (g) = Q(t) \cdot (i^m \hat{f}^m g)$; $\lambda(0) = 1$, $v(0) = \mathbf{1}$ and $\varphi(0) = \hat{\nu}$.

In this case, we have : $Q'(0) = \mathbb{E}_{\hat{\nu}}[i\hat{f}] = 0$ and

$$\lambda'(0) = 0. \tag{3}$$

Moreover, we have $\langle\varphi(0), (Q^n)''(0) \cdot v(0)\rangle_* = \mathbb{E}_{\hat{\nu}}[(S_n(\hat{f}, \hat{T}))^2]$ and hence

$$\lambda''(0) = \lim_{n\to+\infty} \mathbb{E}_{\hat{\nu}}\left[-\left(\frac{S_n(\hat{f}, \hat{T})]}{\sqrt{n}}\right)^2\right] = -\sigma^2(f). \tag{4}$$

Theorem 2.12 (Multidimensional version) *Let $\mathcal{B}$ be a complex Banach space. Let U_0 be an open subset of $\mathbb{R}^\ell$ containing $0_{\mathbb{R}^\ell}$. Let $m \geq 1$ be some integer. Let $(Q(t))_{t\in U_0}$ be a family of continuous linear operators on $\mathcal{B}$ such that :*

(i) The application $t \mapsto Q(t)$ is in $C^m(U_0, \mathcal{L}_{\mathcal{B}})$;

(ii) there exist two subspaces $\mathcal{F}$ and $\mathcal{H}$ of $\mathcal{B}$ such that

(a) $\mathcal{B} = \mathcal{F} \oplus \mathcal{H}$, $Q(0)(\mathcal{F}) \subseteq \mathcal{F}$ and $Q(0)(\mathcal{H}) \subseteq \mathcal{H}$,

(b) $\dim(\mathcal{F}) = 1$ and $Q(0)_{|\mathcal{F}} \equiv id_{|\mathcal{F}}$,

(c) the spectral radius of $Q(0)_{|\mathcal{H}}$ is strictly less than 1.

Then there exists an open set U_1 containing 0 and contained in U_0, there exist three real numbers $\eta_1 > 0, \eta_2 > 0$, $c_1 \geq 0$ and four functions $\lambda \in C^m(U_1, \mathbb{C})$, $v \in C^m(U_1, \mathcal{B})$, $\varphi \in C^m(U_1, \mathcal{B}')$ and $N \in C^m(U_1, \mathcal{L}_{\mathcal{B}})$ such that, for all $t \in U_1$, we have :

(1) for all $n \geq 1$, $Q(t)^n = \lambda(t)^n v(t) \otimes_* \varphi(t) + N(t)^n$,

(2) $Q(t)v(t) = \lambda(t)v(t)$, $Q(t)^*\varphi(t) = \lambda(t)\varphi(t)$ and $\langle\varphi(t), v(t)\rangle_* = 1$,

(3) $|\lambda(t)| \geq 1 - \eta_1$,

(4) for all $\ell = 0, ..., m$, for all $i_1, ..., i_\ell \in \{1, ..., \ell\}$ and all $n \geq 1$, $\left\|\frac{\partial^\ell}{\partial t_{i_1}\cdots\partial t_{i_\ell}}(N(t)^n)\right\|_{\mathcal{L}_{\mathcal{B}}} \leq c_1(1-\eta_1-\eta_2)^n$.

Idea of the proof. This is the multidimensional version of theorem IV-8 of [10] which is based on the theorem of implicit functions (see chapter XIV of [10]). Hence, it is easily extendible to the multidimensional case, *qed.*

We will use theorem 2.12 to prove our theorem 2. We will apply it to $Q(t) = \hat{P}_t$ and $\mathcal{B} = \mathcal{V}_{(\beta,\varepsilon)}$. We have $\lambda(0_{\mathbb{R}^\ell}) = 1$, $v(0_{\mathbb{R}^\ell}) = \mathbf{1}$ and $\varphi(0_{\mathbb{R}^\ell}) = \hat{\nu}$.

Let $h = (h_1, ..., h_\ell) \in \mathbb{R}^\ell$. We can apply theorem 2.11 to $(\tilde{Q}(s) = Q(sh_1, ..., sh_\ell))_{s\in I}$ (for some small interval I containing 0). From formula (3), we get : $\langle\nabla\lambda(0), (h_1, ..., h_\ell)\rangle = 0$ Hence we have : $\nabla\lambda(0) = 0$. From formula (4), we get : $\sum_{j,j'=1}^{\ell} h_j h_{j'} \frac{\partial^2}{\partial t_j \partial t_{j'}}\lambda(0) = -\sigma^2(h_1 f_1 + ... + h_\ell f_\ell) = -\langle\Sigma^2(f)\cdot h, h\rangle$. Hence we have : $\mathrm{Hess}\lambda(0) = -\Sigma^2(f)$.

3 End of the proofs

In this section, we write $\lambda_t := \lambda(t)$, $\varphi_t := \varphi(t)$ and $v_t := v(t)$, where λ, φ and v are the functions given in theorems 2.11 or 2.12 applied to $Q(t) = \hat{P}_t$ and $\mathcal{B} = \mathcal{V}_{(\beta,\varepsilon)}$. To finish, it suffices to prove :

- For theorem 1, that there exists some $\beta > 0$ such that we have :

$$\int_{-\beta\sqrt{n}}^{\beta\sqrt{n}} \frac{\left|\varphi_{\frac{S_n(f,\hat{T})}{\sqrt{n}}}(t) - \varphi_Z(t)\right|}{|t|}\, dt = O(n^{-\frac{1}{2}}).$$

- For theorem 2, that there exists some $\beta > 0$ such that we have :

$$\left(\int_{|t|_\infty < \beta\sqrt{n}} \sum_{k=0}^{\lfloor\frac{\ell}{2}\rfloor+1} \sum_{\{i_1,\dots,i_k\}\in\{1,\dots,\ell\}^k} \left|\frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}}\left(\varphi_{\frac{S_n(f,\hat{T})}{\sqrt{n}}} - \varphi_Z\right)(t)\right|^2\right)^{\frac{1}{2}} = O(n^{-\frac{1}{2}}).$$

3.1 Theorem 1

To conclude, we just have to follow the proof of theorem B of [10] page 12 (cf. section VI-3, cf. also theorem B* of [10] page 84). Let $c_2 > 0$ and $\beta > 0$ be two real numbers such that the closed ball $[-\beta;\beta]$ is contained in I_1 and such that for any $t \in [-\beta;\beta]$, we have $|\lambda_t| \le e^{-c_2 t^2}$ and $e^{-\frac{\sigma^2(f)t^2}{2}} \le e^{-c_2 t^2}$ (this is possible because $\sigma^2(f) > 0$ and because we have $\lambda''(0) = -\sigma^2(f)$). In the following, n will be any integer and $t \in \mathbb{R}$ any real number satisfying : $n \ge 2$ and $|t| < \beta\sqrt{n}$. For such a couple (n,t), we have : $\frac{t}{\sqrt{n}} \in I_1$ and we have :

$$\mathbb{E}_{\hat{H}\,\hat{\nu}}\left[e^{it\frac{S_n(\hat{f},\hat{T})}{\sqrt{n}}}\right] - e^{-\frac{\sigma^2(f)t^2}{2}} = \left\langle \hat{\nu}, \left(\hat{P}_{\frac{t}{\sqrt{n}}}\right)^n (\hat{H})\right\rangle_* - e^{-\frac{\sigma^2(f)t^2}{2}}$$

$$= \left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n \left\langle \hat{\nu}, \left(v_{\frac{t}{\sqrt{n}}} \otimes \varphi_{\frac{t}{\sqrt{n}}}\right)(\hat{H})\right\rangle_* + \left\langle \hat{\nu}, \left(N_{\frac{t}{\sqrt{n}}}\right)^n (\hat{H})\right\rangle_* - e^{-\frac{\sigma^2(f)t^2}{2}}$$

$$= \left[\left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n - e^{-\frac{\sigma^2(f)t^2}{2}}\right] + \left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n \left(\left\langle \hat{\nu}, \left(v_{\frac{t}{\sqrt{n}}} \otimes_* \varphi_{\frac{t}{\sqrt{n}}}\right)(\hat{H})\right\rangle_* - 1\right) + \left\langle \hat{\nu}, \left(N_{\frac{t}{\sqrt{n}}}\right)^n (\hat{H})\right\rangle_*$$

$$= O\left(\frac{1}{\sqrt{n}}|t|_\infty^3 e^{-\frac{c_2 t^2}{2}}\right) + O\left(\frac{|t|_\infty}{\sqrt{n}} e^{-\frac{c_2 t^2}{2}}\right) + c_1 (1-\eta_1-\eta_2)^n \frac{|t|_\infty}{\sqrt{n}} \|\hat{H}\|_{\mathcal{V}_{\beta,\varepsilon}}.$$

Therefore, we have :

$$\left(\int_{|t|<\beta\sqrt{n}} \left|\mathbb{E}_{\hat{H}\,\hat{\nu}}\left[e^{i\frac{tS_n(\hat{f},\hat{T})}{\sqrt{n}}}\right] - e^{-\frac{\sigma^2(f)t^2}{2}}\right|^2 dt\right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{n}}\right).$$

3.2 Theorem 2

To conclude we follow the proof of theorem 2.2.1 of [14]. It is inspired by the previous section.

Let $c_2 > 0$ and $\beta > 0$ be two real numbers such that the closed ball $\bar{B}_{|\cdot|_\infty}(0,\beta)$ is contained in U_1 and such that for any $t \in \bar{B}_{|\cdot|_\infty}(0,\beta)$, we have $|\lambda_t| \le e^{-c_2\langle t,t\rangle}$ and $e^{-\frac{1}{2}\langle t,\Sigma^2(f)t\rangle} \le e^{-c_2\langle t,t\rangle}$ (this is possible because $\Sigma^2(f)$ is invertible and because we have $\mathrm{Hess}\lambda(0) = -\Sigma^2(f)$). In the following, n will be any integer and $t \in \mathbb{R}^\ell$ any real number satisfying : $n \ge 2$ and $|t|_\infty < \beta\sqrt{n}$. For such a couple (n,t), we have : $\frac{t}{\sqrt{n}} \in U_1$. Hence, we have :

$$\begin{aligned}\mathbb{E}_{\hat{H}\cdot\hat{\nu}}\left[e^{i\langle t,\frac{1}{\sqrt{n}}S_n(\hat{f},\hat{T})\rangle}\right] &= \left\langle \hat{\nu}, \left(\hat{P}_{\frac{t}{\sqrt{n}}}\right)^n (\hat{H})\right\rangle_* \\ &= \left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n \left\langle \hat{\nu}, \left(v_{\frac{t}{\sqrt{n}}} \otimes_* \varphi_{\frac{t}{\sqrt{n}}}\right)(\hat{H})\right\rangle_* + \left\langle \hat{\nu}, \left(N_{\alpha,\frac{t}{\sqrt{n}}}\right)^n (\hat{H})\right\rangle_*.\end{aligned}$$

1. We start by giving an estimation when $k=0$. As in the case of theorem 1, we get :

$$\left(\int_{|t|_\infty<\beta\sqrt{n}} \left|\mathbb{E}_{\hat{H}\cdot\hat{\nu}}\left[e^{i\langle t,\frac{1}{\sqrt{n}}S_n(\hat{f},\hat{T})\rangle}\right] - e^{-\frac{1}{2}\langle t,\Sigma^2(f)t\rangle}\right|^2 dt\right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{n}}\right).$$

2. Let k be an integer satisfying $1 \le k \le \left\lfloor \frac{\ell}{2}\right\rfloor + 1$ and $(i_1,...,i_k) \in \{1,...,\ell\}^k$. According to theorem 2.12, we have :

$$\frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}}\mathbb{E}_{\hat{H}\cdot\hat{\nu}}\left[e^{i\langle t,\frac{1}{\sqrt{n}}S_n(\hat{f},\hat{T})\rangle}\right] =$$

$$= \left(\frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}}\left(\left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n\right)\right)\left\langle \hat{\nu}, \left(v_{\frac{t}{\sqrt{n}}} \otimes_* \varphi_{\frac{t}{\sqrt{n}}}\right)(\hat{H})\right\rangle_* + O\left((1+|t|_\infty^k)\frac{e^{-\frac{c_2}{2}\langle t,t\rangle}}{\sqrt{n}}\right) +$$

$$+\frac{1}{n^{\frac{k}{2}}}\left\langle \hat{\nu}, \frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}}(N_\cdot)^n\Big|_{\frac{t}{\sqrt{n}}}(\hat{H})\right\rangle_*$$

$$= \left(\frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}}\left(\left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n\right)\right) + O\left((1+|t|_\infty{}^{k+1})\frac{e^{-\frac{c_2}{2}\langle t,t\rangle}}{\sqrt{n}}\right) + \frac{c_1\,(1-\eta_1-\eta_2)^n}{n^{\frac{k}{2}}}\|\hat{H}\|_{\mathcal{V}_{\beta,\varepsilon}},$$

since $\left\langle \nu, \left(v_{\frac{t}{\sqrt{n}}}\otimes_* \varphi_{\frac{t}{\sqrt{n}}}\right)(\hat{H})\right\rangle_* - 1 = O\left(\frac{|t|_\infty}{\sqrt{n}}\right)$ and $\frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}}\left(\left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n\right) = O\left((1+|t|_\infty^k)e^{-\frac{c_2}{2}\langle t,t\rangle}\right)$.
Now we have to estimate the following quantity :

$$\frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}}\left(\left(\lambda_{\frac{t}{\sqrt{n}}}\right)^n\right) - \frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}} e^{-\frac{1}{2}\langle t,\Sigma^2(f)t\rangle}.$$

In the following $b : \bar{B}_{|\cdot|_\infty}(0,\beta) \to \mathbb{C}$ will be a function $C^{\lfloor\frac{\ell}{2}\rfloor+1}$ on $\bar{B}_{|\cdot|_\infty}(0,\beta)$ such that $b(0)=1$ and $\frac{\partial b}{\partial t_i}(0) = 0$ and $\mathrm{Hess}\, b(0) = -\Sigma^2(f)$ and $|b(t)| \le e^{-c_2\langle t,t\rangle}$ (we will take $b(t) := \lambda_t$ and $b(t) := e^{-\frac{1}{2}\langle t,\Sigma^2(f)t\rangle}$). We have :

$$\frac{\partial^k}{\partial t_{i_1}...\partial t_{i_k}}\left(\left(b\left(\frac{t}{\sqrt{n}}\right)\right)^n\right) = \sum_{\mathcal{A}=\{A_1,...,A_m\}\in Q_k} g_n(\mathcal{A},b)(t),$$

where Q_k is the set of partitions $\mathcal{A}=\{A_1,...,A_m\}$ of $\{1,...,k\}$ in nonempty subsets $A_i = \{l_1^{(i)},...,l_{\#A_i}^{(i)}\}$. and with, for all $\mathcal{A}=\{A_1,...,A_m\}\in Q_k$:

$$g_n(\mathcal{A},b)(t) := n(n-1)...(n-m+1)\left(b\left(\frac{t}{\sqrt{n}}\right)\right)^{n-m}\prod_{i=1}^m\left(\frac{\partial^{\#A_i}b}{\partial t_{i_{l_1^{(i)}}}...\partial t_{i_{l_{\#A_i}^{(i)}}}}\right)\left(\frac{t}{\sqrt{n}}\right)n^{-\frac{k}{2}}.$$

Let $\mathcal{A}=\{A_1,...,A_m\}\in Q_k$. $m_0(\mathcal{A})$ will be the number of $A_i\in\mathcal{A}$ such that $\#A_i = 1$. Let us notice that we have $2m \le m_0(\mathcal{A})+k$ and :

$$\begin{aligned}
|g_n(\mathcal{A},b)(t)| &\le n^m e^{-\frac{c_2(n-m)}{n}\langle t,t\rangle} O\left(\left(\frac{|t|_\infty}{\sqrt{n}}\right)^{m_0(\mathcal{A})}\right) n^{-\frac{k}{2}}\\
&= O\left(n^{\frac{1}{2}(2m-(m_0(\mathcal{A})+k))}|t|_\infty{}^{m_0(\mathcal{A})}e^{-\frac{c_2}{2}\langle t,t\rangle}\right)\\
&= O\left(|t|_\infty{}^{m_0(\mathcal{A})}e^{-\frac{c_2}{2}\langle t,t\rangle}\right).
\end{aligned}$$

- If $2m < m_0(\mathcal{A})+k$, then, for any $t\in B_{|\cdot|_\infty}(0,\beta\sqrt{n})$, we have :

$$|g_n(\mathcal{A},b)(t)| = O\left(\frac{|t|_\infty{}^{m_0(\mathcal{A})}}{\sqrt{n}}e^{-\frac{c_2}{2}\langle t,t\rangle}\right).$$

- If $2m = m_0(\mathcal{A})+k$, each A_j has cardinality one or two. Hence, for any $t\in B_{|\cdot|_\infty}(0,\beta\sqrt{n})$, we have :

$$\left|g_n(\mathcal{A},\lambda_\cdot)(t) - g_n(\mathcal{A},e^{-\frac{1}{2}\langle\cdot,\Sigma^2(f)\cdot\rangle})(t)\right| = O\left(\frac{1}{\sqrt{n}}(1+|t|_\infty^{m_0(\mathcal{A})+3})e^{-\frac{c_2}{2}\langle t,t\rangle}\right).$$

Indeed, we have :

$$\frac{\partial}{\partial t_i}\left(\lambda_\cdot - e^{-\frac{1}{2}\langle\cdot,\Sigma^2(f)\cdot\rangle}\right)\left(\frac{t}{\sqrt{n}}\right) = O\left(\frac{|t|_\infty{}^2}{n}\right)$$

and

$$\left(\lambda_{\frac{t}{\sqrt{n}}}\right)^{n-m} - e^{-\frac{n-m}{2n}\langle t,\Sigma^2(f)t\rangle} = O\left(\frac{|t|_\infty^3}{\sqrt{n}}e^{-\frac{c_2}{2}\langle t,t\rangle}\right)$$

and

$$\frac{\partial^2}{\partial t_i\partial t_j}\left(\lambda_\cdot - e^{-\frac{1}{2}\langle\cdot,\Sigma^2(f)\cdot\rangle}\right)\left(\frac{t}{\sqrt{n}}\right) = O\left(\frac{|t|_\infty}{\sqrt{n}}\right),$$

qed.

3.3 Theorem 3

Let F be as in the hypothesis of theorem 3.

1. Let us notice that :

$$(\mu_1)_*\left(\frac{1}{\sqrt{t}}\int_0^t F\circ Z_s\,ds\right)=\mu_*\left(\frac{1}{\sqrt{t}}\int_0^t F\circ\psi^{-1}\circ Y_s\,ds\right).$$

2. For any $\omega\in M$ and any $t>0$, we define the number $n(t,\omega)$ of collisions before the time t for a particle having configuration ω at time 0 :

$$n(t,\omega):=\max\left\{k\geq 0\ :\ \sum_{j=0}^{k-1}\tau\circ T^j(\omega)\leq t\right\}.$$

Let us define $f(\omega):=\int_0^{\tau(\omega)}F\circ\psi^{-1}(\omega,s)\,ds$. Let us denote by $\mathcal{K}_\mu$ the Ky-Fan metric for $\mathbb{R}^\ell$-random variables defined on $\mathcal{M}$. Since $Y_s(\omega,u)=Y_{s+u}(\omega)$ and since $Y_{\sum_{j=0}^{k-1}\tau\circ T^j(\omega)}(\omega,0)=T^k(\omega)$ and since

$$\sum_{k=0}^{n(t,\omega)-1}f\circ T^k(\omega)=\int_0^{\sum_{j=0}^{n(t,\omega)-1}\tau\circ T^j(\omega)}(F\circ\psi^{-1})\circ Y_s(\omega,0)\,ds,$$

we have :

$$\left\|\frac{1}{\sqrt{t}}\int_0^t(F\circ\psi^{-1})\circ Y_s ds-\frac{1}{\sqrt{t}}\sum_{k=0}^{n(t,q(\cdot))-1}f\circ T^k(q(\cdot))\right\|_{L^\infty(\mu)}=O\left(t^{-\frac{1}{2}}\right),$$

with $q:(\omega,s)\mapsto\omega$. Hence we have :

$$\mathcal{K}_\mu\left(\frac{1}{\sqrt{t}}\int_0^t(F\circ\psi^{-1})\circ Y_s ds,\frac{1}{\sqrt{t}}\sum_{k=0}^{n(t,q(\cdot))-1}f\circ T^k(q(\cdot))\right)=O\left(t^{-\frac{1}{2}}\right),$$

3. Moreover we have :

$$\tilde{\mu}_*\left(\frac{1}{\sqrt{t}}\sum_{k=0}^{n(t,q(\cdot))-1}f\circ T^k(q(\cdot))\right)=(\tau\cdot\nu)_*\left(\frac{1}{\sqrt{t}}\sum_{k=0}^{n(t,\cdot)-1}f\circ T^k(\cdot)\right).$$

4. Let us write $\bar{\tau}:=\int_M\tau\,d\nu$. According to theorem 2, since the coordinates of f belong to $\mathcal{H}_{(\eta,1)}$ and are ν-centered, we have :

$$\Pi\left((\tau\cdot\nu)_*\left(\frac{1}{\sqrt{t}}\sum_{k=0}^{\lfloor\frac{t}{\bar{\tau}}\rfloor-1}f\circ T^k(\cdot)\right),\mathcal{N}\left(0,\frac{1}{\bar{\tau}}\Sigma^2(f)\right)\right)=O\left(\frac{1}{\sqrt{t}}\right).$$

5. We are led to the control of $W_t:=\frac{1}{\sqrt{t}}\left(\sum_{k=0}^{n(t,\cdot)-1}f\circ T^k-\sum_{k=0}^{\lfloor\frac{t}{\bar{\tau}}\rfloor-1}f\circ T^k\right)$.

6. According to lemma 4.1 of [13], we have :

$$\forall L>0,\ \ \exists C_L>0,\ \forall t>1,\ \forall K\geq 1,\ \nu\left(\left\{\left|n(t,\cdot)-\frac{t}{\bar{\tau}}\right|\geq K\sqrt{t}\right\}\right)\leq C_L K^{-L}. \qquad (5)$$

7. Moreover $(\sum_{k=0}^{n-1} f\circ T^k)_k$ and $(\sum_{k=0}^{n-1}(\tau\circ T^k-\bar\tau))_k$ are bounded in all L^p. Hence, for all $p\geq 1$, there exists $A_p>0$ such that :

$$\forall t>0,\quad \left\| \frac{1}{\sqrt{t}}\left((\sum_{k=0}^{\lfloor \frac{t}{\bar\tau}\rfloor -1} \tau\circ T^k) - t\right)\right\|_{L^p(\nu)} \leq A_p. \tag{6}$$

And, according to a result of Serfling (theorem B of [15]), for all $p\geq 1$ and there exists $K_p>0$ such that we have :

$$\forall N\geq 1,\quad \left\| \max_{n=0,\dots,N} (\max\left(|S_n(f,T)|, |S_n(f,T^{-1})|\right))\right\|_{L^p(\nu)} \leq K_p N^{\frac{1}{2}}. \tag{7}$$

8. Let $\alpha\in\left]0;\frac{1}{4}\right[$. According to (7), we have :

$$\forall p\geq 1,\quad \forall t>0,\quad \left\| W_t \mathbf{1}_{\{|n(t,\cdot)-\frac{t}{\bar\tau}|\leq t^{\frac{1}{2}+2\alpha}\}}\right\|_{L^p(\nu)} \leq 2K_p\frac{t^{\frac{1}{4}+\alpha}}{\sqrt{t}}.$$

Moreover, according to (5) and to (6) and to the Cauchy-Schwartz inequality, we have :

$$\forall p\geq 1,\quad \forall L\geq 1,\ \forall t>0,\quad \left\| W_t \mathbf{1}_{\{|n(t,\cdot)-\frac{t}{\bar\tau}|> t^{\frac{1}{2}+2\alpha}\}}\right\|_{L^p(\nu)} \leq (A_{2p}+\|\tau\|_\infty)\|F\|_\infty C_L{}^{\frac{1}{2p}} t^{-2\alpha L}.$$

9. By taking $L:=\frac{\frac{1}{4}-\alpha}{2\alpha}$, we have :

$$\forall \alpha>0,\ \forall p\geq 1,\ \sup_{t>\bar\tau} t^{\frac{1}{4}-\alpha}\|W_t\|_{L^p(\nu)}<+\infty \quad\text{and}\quad \sup_{t>\bar\tau} t^{\frac{1}{4}-\alpha}\|W_t\|_{L^p(\tau\cdot\nu)}<+\infty.$$

From this and from the Markov inequality, we get :

$$\forall\alpha>0,\ \forall p\geq 1,\quad \mathcal{K}_{\tau\cdot\nu}\left(\frac{1}{\sqrt{t}}\sum_{k=0}^{\lfloor\frac{t}{\bar\tau}\rfloor-1} f\circ T^k, \frac{1}{\sqrt{t}}\sum_{k=0}^{n(t,\cdot)-1} f\circ T^k(\cdot)\right) = O\left(t^{-\frac{p}{p+1}\left(\frac{1}{4}-\alpha\right)}\right).$$

Let us notice that it seems difficult to get a better result than $O\left(t^{-\frac{1}{4}}\right)$ with this method. Indeed $\left(\frac{n(t,\cdot)-\frac{t}{\bar\tau}}{\sqrt{t}}\right)_t$ converges in distribution (when t goes to infinity) to a non-degenerate gaussian random variable. Hence, we suspect that $\frac{1}{\sqrt{t}}\left(\sum_{k=0}^{\lfloor\frac{t}{\bar\tau}\rfloor-1} f\circ T^k - \sum_{k=0}^{n(t,\cdot)-1} f\circ T^k(\cdot)\right)$ is of order $t^{-\frac{1}{4}}$. But this does not preclude a rate of convergence in $t^{-\frac{1}{2}}$ for the $\frac{1}{\sqrt{t}}\sum_{k=0}^{n(t,\cdot)-1} f\circ T^k(\cdot)$.

References

[1] A. C. Berry, *The accuracy of the Gaussian approximation to the sum of independent variates*, Trans. Amer. Math. Soc. 49, 122–136 (1941).

[2] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture Notes in Mathematics 470, Springer-Verlag, 108 p. (1975)

[3] L.A. Bunimovich & Ya.G. Sinai & N.I. Chernov, *Statistical properties of two-dimensional hyperbolic billiards*, Russ. Math. Surv. 46, No.4, 47-106 (1991); translation from Usp. Mat. Nauk 46, No.4(280), 43-92 (1991).

[4] Bunimovich, L.A.; Sinai, Ya.G, *Statistical properties of Lorentz gas with periodic configuration of scatterers*, Commun. Math. Phys. 78, 479-497 (1981)

[5] N. I. Chernov, *Decay of correlation and dispersing billiards*, J. Stat. Phys., vol. 94 (3/4), 513–556 (1999).

[6] R. M. Dudley, *Real Analysis and Probability*, Wadsworth & Brooks Cole Math. series (1989).

[7] C. Esseen, *Fourier Analysis of distribution functions. A mathematical study of the Laplace-Gaussian Law*, Acta Math., vol. 77 (1945), 1–125.

[8] S. Gouëzel, *Berry-Esseen theorem and local limit theorem for non uniformly expanding maps*, Ann. Inst. H. Poincaré Probab. Statist. 41 (2005), 997–1024.

[9] Y. Guivarc'h & J. Hardy, *Théorèmes limites pour une classe de chaînes de Markov et applications aux difféomorphismes d'Anosov* , Ann. Inst. H. Poincaré Probab. Statist. 24 (1988), no. 1, 73–98.

[10] H. Hennion & L. Hervé, *Limit theorems for Markov Chains and Stochastic Properties of Dynamical Systems by Quasi-Compactness*, Lecture Notes in Mathematics, vol. 1766, Berlin : springer, 145 p. (2001).

[11] S. V. Nagaev, *Some limit theorems for stationary Markov chains*, Theor. Probab. Appl. 2, 378–406 (1957) translation from Teor. Veroyatn. Primen. 2, 389-416 (1958).

[12] S. V. Nagaev, *More exact statement of limit theorems for homogeneous Markov chains*, Theor. Probab. Appl. 6, 62-81 (1961); translation from Teor. Veroyatn. Primen, 6, 67-86 (1961).

[13] F. Pène, *Rates of convergence in the CLT for two-dimensional dispersive billiards*, Commun. Math. Phys. 225, No.1, 91-119 (2002).

[14] F. Pène, *Rate of convergence in the multidimensional CLT for stationary processes. Application to the Knudsen gas and to the Sinai billiard*, accepted for publication in Annals of Applied Probability.

[15] R. J. Serfling, it Moment inequalities for the maximum cumulative sum, Ann. Math. Stat. 41, 1227-1234 (1970).

[16] Ya.G. Sinai, *Dynamical systems with elastic reflections*, Russ. Math. Surv. 25, No.2, 137-189 (1970)

[17] D. Szász & T. Varjú *Local limit theorem for the Lorentz process and its recurrence in the plane*, Ergodic Theory Dyn. Syst. 24, No.1, 257-278 (2004).

[18] L.-S. Young, *Statistical properties of dynamical systems with some hyperbolicity. Ann. of Math.*, vol. 147 (1998), 585–650.

[19] V. V. Yurinskii, *A smoothing inequality for estimates of the Levy-Prokhorov distance*, Theory Probab. Appl. 20, 1–10 (1975); translation from Teor. Veroyatn. Primen. 20, 3-12 (1975).